

BORCHARDT'S FORM OF THE ELIMINANT OF TWO EQUATIONS OF THE n th DEGREE.

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1. The problem which Borchardt set for himself was to express the eliminant of the equations $\phi(x)=0$, $\psi(x)=0$, both of the n th degree, in terms of $n+1$ arbitrary values of x and the corresponding values of $\phi(x)$, $\psi(x)$. To this end he made use of the result of Cayley's mode of finding Bezout's condensed eliminant; and as a consequence his solution took the form of an expression in terms of the $\frac{1}{2}n(n+1)$ different values of

$$\frac{\phi(x)\psi(y) - \phi(y)\psi(x)}{y - x}$$

found by giving x and y the said $n+1$ arbitrary values, say, $a_0, a_1, \dots, a_n$. He experienced difficulty, of course, when x and y had to be given one and the same value; but this was overcome by establishing the theorem that *If $\phi(x)$, $\psi(x)$ be rational integral functions of the n th degree, and $\{\phi(x)\psi(y) - \phi(y)\psi(x)\} \div (y-x)$ be denoted by $F(x,y)$ and $(x-a_0)(x-a_1) \dots (x-a_n)$ by $f(x)$, then*

$$\frac{F(a_r, a_r)}{f'(a_r)} = - \sum_s \frac{F(a_r, a_s)}{f'(a_s)}$$

where s is given in succession all the values $0, 1, \dots, n$, except r . In other words, he succeeded in showing that each of the $n+1$ illusory forms becomes known as a consequence of knowing n of the other forms.*

2. The mode of proof given by him is laborious, the two given functions being expressed in the interpolational form

$$\sum_{r=0}^{r=n} \frac{f(x)/(x-a_r)}{f'(a_r)} \phi(a_r), \quad \sum_{r=0}^{r=n} \frac{f(x)/(x-a_r)}{f'(a_r)} \psi(a_r),$$

* For marked advances recently made on Borchardt's work the reader is referred to two papers by Mr. A. L. Dixon in the *Proceed. Lond. Math. Soc.* (2), vi. pp. 468-478, vii. pp. 49-69.

the operations involved in $\phi(x)\psi(y) - \phi(y)\psi(x)$ then performed, and the terms of the result combined in groups of four so as to admit of the factor $y - x$ being struck out preparatory to making the substitution $x = y = a_r$.

A much simpler mode would have been to take $\phi(x)\psi'(x) - \phi'(x)\psi(x)$ as the limiting form of $\{\phi(x)\psi(y) - \phi(y)\psi(x)\} \div (y - x)$, and then substitute for $\phi'(a_r)$, $\psi'(a_r)$ the expressions obtained from the interpolational forms of $\phi(x)$, $\psi(x)$ by dividing by $x - a_r$ and thereafter putting $x = a_r$.

3. As is often the case, however, the whole matter suffers as regards simplicity by reason of excessive specialisation. The following much more general theorem is susceptible of a much simpler proof. If $\Phi(x, y)$ be used to stand for

$$\begin{array}{cccc|c} 1 & x & \dots & x^{n-1} & \\ a_{11} & a_{12} & \dots & a_{1n} & 1 \\ a_{21} & a_{22} & \dots & a_{2n} & y \\ \dots & \dots & \dots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} & y^{n-1}, \end{array}$$

and $\zeta^{\frac{1}{2}}$ for the difference-product of any quantities following it, then

$$\zeta^{\frac{1}{2}}(a_1 a_2 \dots a_n) \cdot \Phi(a_0 a_0) - \zeta^{\frac{1}{2}}(a_0 a_2 \dots a_n) \cdot \Phi(a_0 a_1) + \zeta^{\frac{1}{2}}(a_0 a_1 a_3 \dots a_n) \Phi(a_0 a_2) - \dots + (-1)^n \zeta^{\frac{1}{2}}(a_0 a_1 \dots a_{n-1}) \cdot \Phi(a_0 a_n) = 0.$$

By way of proof we have only to seek for the co-factor of a_{rs} on the left-hand side. Now in $\Phi(x, y)$ this co-factor is seen to be $y^{r-1} x^{s-1}$, therefore the full co-factor sought is

$$\zeta^{\frac{1}{2}}(a_1 a_2 \dots a_n) \cdot a_0^{r-1} a_0^{s-1} - \zeta^{\frac{1}{2}} a_0 a_2 \dots a_n) \cdot a_0^{r-1} a_1^{s-1} + \zeta^{\frac{1}{2}}(a_0 a_1 a_3 \dots a_n) \cdot a_0^{r-1} a_2^{s-1} - \dots + (-1)^n \zeta^{\frac{1}{2}}(a_0 a_1 \dots a_{n-1}) \cdot a_0^{r-1} a_n^{s-1}$$

which is seen to be the development of

$$a_0^{r-1} (-1)^n \begin{vmatrix} 1 & a_0 & a_0^2 & \dots & a_0^{n-1} & a_0^{s-1} \\ 1 & a_1 & a_1^2 & \dots & a_1^{n-1} & a_1^{s-1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & a_n & a_n^2 & \dots & a_n^{n-1} & a_n^{s-1} \end{vmatrix}$$

arranged according to the elements of the last column of the determinant. As, however, s is not greater than n , the said last column must be identical with a preceding column; and thus the theorem is proved.

Borchardt's special case of this is where the square array in the expression of $\Phi(x, y)$ is the peculiar axisymmetric array of Bezout's condensed eliminant, for then by Cayley's theorem $\Phi(x, y)$ becomes

$$\{\phi(x)\psi(y) - \phi(y)\psi(x)\} \div (y - x) \quad \text{or} \quad F(x, y).$$

This is virtually the same as saying that *If the sum of every row, except one, of a determinant be 0, the co-factors of the elements of the excepted row are identical.* Consequently we can assert that *Any determinant which has the sum of every row equal to zero has in the case of every row the same co-factor for every element of the row ; and from this it follows at once that Any axisymmetric determinant which has the sum of every row equal to zero has the co-factors of all the elements identical*—which is Borchardt's theorem.

5. It is very doubtful, however, whether it is desirable to view this common co-factor as a determinant at all ; for when it is of the n th order it is a function of only $\frac{1}{2}n(n+1)$ elements and yet contains $(n+1)^{n-1}$ terms and all of them positive. Thus

$$\begin{vmatrix} a_2 + b_3 + b_4 & -b_3 & -b_4 \\ -b_3 & a_3 + b_3 + c_4 & -c_4 \\ -b_4 & -c_4 & a_4 + b_4 + c_4 \end{vmatrix},$$

which is a function of the six quantities $a_2, a_3, a_4, b_3, b_4, c_4$, is equal to

$$\begin{aligned} & a_2 a_3 a_4 + a_2 a_3 (b_4 + c_4) + a_2 a_4 (b_3 + c_4) + a_3 a_4 (b_3 + b_4) \\ & + (a_2 + a_3 + a_4)(b_3 b_4 + b_3 c_4 + b_4 c_4). \end{aligned}$$

Further, although it is of the third order, there are not two but three other ways of writing it like this ; and, generally, $n+1$ ways in all in the case of the n th order—a property unnatural to a determinant. The reason, of course, is that these $n+1$ ways correspond to the co-axial primary minors of the determinant of the $(n+1)$ th order whose primary minors are all equal.

6. Much more convenience results from arranging the $\frac{1}{2}n(n+1)$ elements in semi-quadrant form, that is to say, in the manner in which the elements of a pfaffian are arranged ; and if after doing this we bracket them in some specially distinctive way, we shall at the same time secure a convenient notation. Thus, we shall have

$$\left| \begin{array}{ccc} \{ a_2 & a_3 & a_4 \\ & b_3 & b_4 \\ & & c_4 \} \end{array} \right|$$

instead of the determinant in the preceding paragraph, and

$$\left| \begin{array}{cc} \{ a_2 & a_3 \\ & b_3 \} \end{array} \right| \quad \text{instead of} \quad \begin{vmatrix} a_2 + b_3 & -b_3 \\ -b_3 & a_3 + b_3 \end{vmatrix}.$$

Further, towards the formulating of a recurrent law of formation for the new function we shall have the following identities :—

$$|\{a_2\}| = a_2, \quad |\{a_2 \quad a_3 \atop b_3\}| = (a_2 + a_3)|\{b_3\}| + a_2a_3,$$

$$|\{a_2 \quad a_3 \quad a_4 \atop b_3 \quad b_4 \atop c_4\}| = (a_2 + a_3 + a_4)|\{b_3 \quad b_4 \atop c_4\}| + a_2a_3|\{b_4 + c_4\}| + a_2a_4|\{b_3 + c_4\}| + a_3a_4|\{b_3 + b_4\}| + a_2a_3a_4,$$

$$|\{a_2 \quad a_3 \quad a_4 \quad a_5 \atop b_3 \quad b_4 \quad b_5 \atop c_4 \quad c_5 \atop d_5\}| = (a_2 + a_3 + a_4 + a_5)|\{b_3 \quad b_4 \quad b_5 \atop c_4 \quad c_5 \atop d_5\}| + \Sigma a_2a_3|\{b_4 + c_4 \quad b_5 + c_5 \atop d_5\}| + \Sigma a_2a_3a_4|\{b_5 + c_5 + d_5\}| + a_2a_3a_4a_5,$$

and so forth. With the double-suffix notation for the elements, or with the umbral notation, the details of the law are still more readily grasped. Thus the next identity of the series may be written

$$|\{1_2 \quad 1_3 \dots 1_6 \atop 2_3 \dots 2_6 \atop \dots \atop 5_6\}| = 1_2 + 1_3 + \dots + 1_6) |\{2_3 \quad 2_4 \quad 2_5 \quad 2_6 \atop 3_4 \quad 3_5 \quad 3_6 \atop 4_5 \quad 4_6 \atop 5_6\}| + \Sigma 1_21_3|\{2_4 + 3_4 \quad 2_5 + 3_5 \quad 2_6 + 3_6 \atop 4_5 \quad 4_6 \atop 5_6\}| + \Sigma 1_21_31_4|\{2_5 + 3_5 + 4_5 \quad 2_6 + 3_6 + 4_6 \atop 5_6\}| + \Sigma 1_21_31_41_5(2_6 + 3_6 + 4_6 + 5_6) + 1_21_31_41_51_6,$$

where we observe that by deleting the first element of the co-factor of 1_2 , and then combining the 1st and 2nd frame-lines by addition we obtain the co-factor of 1_21_3 ; and by treating similarly the co-factor of 1_21_3 we obtain the co-factor of $1_21_31_4$: and so on.

7. The development being arranged according to products of elements of the first line, what is required for practical purposes is a rule for telling the co-factor of such a product; and this is easily formulated. Thus, if the co-factor of 1_51_6 be wanted, we note that the indices not found in 1_51_6 are 2, 3, 4, and thence form the triangular array

$$\begin{array}{cc} 23 & 24 \\ & 34, \end{array}$$

thereafter prefixing to this array the line of three elements got from taking 2 along with 5 and 6, 3 along with 5 and 6, and 4 along with 5 and 6, the result being

$$|\{2_5 + 2_6 \quad 3_5 + 3_6 \quad 4_5 + 4_6 \atop 2_3 \quad 2_4 \atop 3_4\}|.$$

Similarly the co-factor of $1_4 1_5 1_6$ is found to be

$$\left| \begin{array}{cc} 2_4 + 2_5 + 2_6 & 3_4 + 3_5 + 3_6 \\ & 2_3 \end{array} \right|.$$

8. The development, however, may also be arranged according to products of elements of any other frame-line than the first. Thus, taking the *third* frame-line $1_3, 2_3, 3_4, 3_5, 3_6$, and seeking the co-factor of $2_3 3_5$, we first note that the indices not found in $2_3 3_5$ are 1, 4, 6, and we thereupon form the triangular array

$$\begin{array}{cc} 14 & 16 \\ & 46, \end{array}$$

to which we next prefix the line of three elements got from taking 1 along with 2 and 5, 4 along with 2 and 5, and 6 along with 2 and 5, the full co-factor sought being

$$\left| \begin{array}{ccc} 1_2 + 1_5 & 2_4 + 4_5 & 2_6 + 5_6 \\ & 1_4 & 1_6 \\ & & 4_6 \end{array} \right|.$$

This suggests that the third, and indeed any, frame-line may be made the first; and we readily find that the procedure necessary to effect the transformation is (1) to delete the third frame-line from the given triangular array, and write it again as the first frame-line of a new array, and (2) to complete the said new array by appending the undeleted minor of the original array, the result being

$$\left| \begin{array}{ccccc} 1_3 & 2_3 & 3_4 & 3_5 & 3_6 \\ & 1_2 & 1_4 & 1_5 & 1_6 \\ & & 2_4 & 2_5 & 2_6 \\ & & & 4_5 & 4_6 \\ & & & & 5_6 \end{array} \right|.$$

Since any frame-line may thus occupy any one of the available places, the number of different modes of writing the function of the n th degree is $(n + 1)!$, the last mode being got from the first by not only reversing the order of the frame-lines but also reversing the order of the elements in each: for example, the first and last modes of writing the function of the 4th degree are

$$\left| \begin{array}{cccc} a_2 & a_3 & a_4 & a_5 \\ & b_3 & b_4 & b_5 \\ & & c_4 & c_5 \\ & & & d_5 \end{array} \right|, \quad \left| \begin{array}{cccc} d_5 & c_5 & b_5 & a_5 \\ & c_4 & b_4 & a_4 \\ & & b_3 & a_3 \\ & & & a_2 \end{array} \right|.$$